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Octic Anharmonic Oscillators: Perturbed Coherent States and the Classical Limit

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Abstract We use the Poincaré-Linstedt method to find a classical perturbation solution to the octic anharmonic oscillator. Next, we derive perturbed coherent states for this system, calculate the expectation value of the $\hat{x}$-operator in them and enforce a limiting process to retrieve the classical result from the corresponding quantum one. We have observed a frequency shift proportional to the sixth power of the amplitude for this system. Our results are in agreement with those obtained from Taylor-series method.

Keywords Octic anharmonic oscillator · Classical limit · Coherent states · Perturbation theory

1 Introduction

Anharmonic oscillators are not only important systems on their own rights, but they are also a testing ground to check and compare a wide variety of numerical and analytical techniques that may be utilized to solve classical and quantum systems [1–12]. The quartic anharmonic oscillator is the archetypal model that has been used over and over to serve this purpose [13–22]. There are also several investigations concerning the higher order and general anharmonic oscillators [23–33]. We mention also a small number of papers using anharmonic oscillators to investigate the classical-quantum connections; the classical limit of the quartic anharmonic oscillator is discussed in [34] and [35], and that of the sextic anharmonic oscillator in [36]. In [37] the reverse path is taken; a Taylor-series method is applied to find the quantum solution from the classical one, considering quartic, sextic and octic anharmonicities.

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In this work we consider an octic anharmonic oscillator whose Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 + \lambda x^8. \tag{1}$$

This system has possible applications in different arias of physics and chemistry [37, 38], but also provides a suitable ground to apply various classical and quantum techniques and to scrutinize their interconnections, untested for such anharmonicity so far. Our intention concerning this Hamiltonian is fourfold: (a) We use the Poincaré-Linstedt method to find the classical solution, (b) using Rayleigh-Schrödinger perturbation theory, we obtain the quantum solution [39]. (c) We construct the perturbed α-coherent states using the quantum solution and calculate the expectation value of the $\hat{x}$-operator in them. (d) We apply the $\hbar \to 0$, $\alpha \to \infty$ limit, to retrieve the classical solution from the quantum one [40]. In conclusion, we shall establish that the sequence of the Rayleigh-Schrödinger perturbation theory, perturbed coherent states and our limiting process, is consistent with the classical results obtained via the Hamilton's equations of motion and the Poincaré-Linstedt method, for the octic anharmonicity. We obtain a first order frequency shift proportional to the sixth power of the amplitude. It is also observed that only cosine terms with coefficients proportional to the seventh power of the amplitude, and frequencies equal to ω_0, $3\omega_0$, $5\omega_0$ and $7\omega_0$ appear in the first order solution. A comparison between our results and others' will be also made, when available.

The organization of the rest of this paper is as follows. We take up the classical solution in Sect. 2. The quantum solution is dealt with in Sect. 3. We set up the coherent states and carry out the classical limit in Sect. 4. Finally, Sect. 5 will be devoted to the conclusions and discussion.

2 Classical Solution

Using the Hamilton's equations of motion, the Hamiltonian (1) leads to the differential equation

$$\ddot{x} + \omega_0^2 x + \frac{8\lambda}{m} x^7 = 0, \tag{2}$$

which we want to solve. We define the scaled time variable as follows

$$s = t(1 + \lambda\omega_1 + \lambda^2\omega_2 + \cdots), \tag{3}$$

where, ω_i's will be determined later. We consider the solution

$$x(s) = \sum_{n=0}^{\infty} \lambda^n x_n(s). \tag{4}$$

Substituting (4) in (2) and putting the coefficients of the powers of λ up to the first order equal to zero we find

$$\frac{d^2 x_0}{ds^2} + \omega_0^2 x_0 = 0, \tag{5}$$

$$\frac{d^2 x_1}{ds^2} + \omega_0^2 x_1 + 2\omega_0^2\omega_1 x_0 + \frac{8}{m} x_0^7 = 0. \tag{6}$$

The first order solution

$$x_0(s) = A' \cos \omega_0 s, \tag{7}$$

satisfies (5) and its substitution in (6) leads to

$$\ddot{x}_1 + \omega_0^2 x_1 + \lambda \frac{A'^7}{8m}[\cos 7\omega_0 s + 21 \cos 3\omega_0 s + 7 \cos 5\omega_0 s] = 0, \tag{8}$$

$$\omega_1 = -\frac{35}{16m\omega_0^2} A'^6, \tag{9}$$

where the following identity has been used

$$\cos^7 x = \frac{1}{64}(35 \cos x + 21 \cos 3x + 7 \cos 5x + \cos 7x). \tag{10}$$

The differential equation (8) now leads to the solution

$$x_1(s) = \frac{A'^7}{384m\omega_0^2}[126 \cos 3\omega_0 s + 14 \cos 5\omega_0 s + \cos 7\omega_0 s - 141 \cos \omega_0 s]. \tag{11}$$

Assuming $\omega_0 s = \omega t$ and substituting (11) and (7) in (4) and using (3) and (9), we find the classical solution up to the first order in the parameter λ as follows

$$x(t) = A' \cos \omega t + \frac{\lambda A'^7}{m\omega_0^2}\left[\frac{21}{64} \cos 3\omega t + \frac{7}{192} \cos 5\omega t + \frac{1}{384} \cos 7\omega t\right], \tag{12}$$

where

$$\omega = \omega_0(1 + \lambda\omega_1 + \cdots)^{-1} = \omega_0\left(1 + \frac{35\lambda A'^6}{16m\omega_0^2} + \cdots\right). \tag{13}$$

Using the initial conditions

$$x(0) = A, \qquad p(0) = m\dot{x}(0) = 0, \tag{14}$$

we finally obtain the classical solution to the octic anharmonic oscillator as follows

$$x(t) = A \cos \omega t + \frac{\lambda A^7}{384m\omega_0^2}[-141 \cos \omega t + 126 \cos 3\omega t + 14 \cos 5\omega t + \cos 7\omega t] + o\lambda^2, \tag{15}$$

where the frequency shift is given by

$$\omega = \omega_0\left(1 + \frac{35\lambda}{16m\omega_0^2} A^6 + \cdots\right). \tag{16}$$

We note that (15) and (16) are consistent with the result given in [37], derived from a Taylor-series-expansion method, if a simple parameter adjustment is made. We also observe that the frequency shift is proportional to the sixth power of the amplitude, as one could anticipate from the already observed trend in the case of quartic [34, 35] and sextic [36] anharmonicities; their frequency shift is proportional to the second and the fourth order of the amplitude, respectively.

3 Quantum Solution

We are going to solve the Schrödinger equation

$$H|\psi_n\rangle = E_n|\Psi_n\rangle, \tag{17}$$

where

$$H = H_0 + \lambda x^8 = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 + \lambda x^8. \tag{18}$$

We also assume

$$H_0|n\rangle = E_0^n|n\rangle, \tag{19}$$

$$\langle m|n\rangle = \delta_{mn}. \tag{20}$$

Using the Rayleigh-Schrödinger perturbation theory [39] we may write

$$|\psi_n\rangle = |\psi_n^0\rangle + \lambda|\psi_n^1\rangle + \cdots, \tag{21}$$

where

$$|\psi_n^0\rangle = |n\rangle, \tag{22}$$

$$|\psi_n^1\rangle = \sum_{k\neq n} \frac{\langle k|x^8|n\rangle}{E_n^0 - E_k^0}|k\rangle. \tag{23}$$

We note that the set $\{|\psi_n\rangle\}$ is orthonormal up to the first order in the parameter λ.

Writing the operator $\hat{x}$ in terms of the creation and annihilation operators

$$\hat{x} = \frac{1}{\sqrt{2m\omega}}(a + a^+), \tag{24}$$

we can show

$$\begin{aligned}\langle k|\lambda x^8|n\rangle = \lambda\left(\frac{\hbar}{2m\omega_0}\right)^4 &[\sqrt{(n+1)\cdots(n+8)}\langle k|n+8\rangle \\ &+ (8n+28)\sqrt{(n+1)\cdots(n+6)}\langle k|n+6\rangle \\ &+ (28n^2+140n+210)\sqrt{(n+1)\cdots(n+4)}\langle k|n+4\rangle \\ &+ (56n^3+252n^2+532n+420) \\ &\times\sqrt{(n+1)(n+2)}\langle k|n+2\rangle + \sqrt{(n-1)\cdots(n-8)}\langle k|n-8\rangle \\ &+ (8n-20)\sqrt{n(n-1)\cdots(n-5)}\langle k|n-6\rangle \\ &+ (28n^2-84n+98)\sqrt{n(n-1)\cdots(n-3)}\langle k|n-4\rangle \\ &+ (56n^3-84n^2-196n-84)\sqrt{n(n-1)}\langle k|n-2].\end{aligned} \tag{25}$$

Now, using (25) in (23) we find the perturbed wave function $|\psi_n\rangle$ up to the first order in the parameter λ as follows

$$
\begin{aligned}
|\psi_n\rangle = |n\rangle + \frac{\lambda\hbar^4}{\hbar\omega_0(2m\omega_0)^4}\Bigg[& -\frac{1}{8}\sqrt{(n+1)\cdots(n+8)}|n+8\rangle \\
& -\frac{1}{6}(8n+28)\sqrt{(n+1)\cdots(n+6)} \\
& \times |n+6\rangle - \frac{1}{4}(28n^2+140n+210)\sqrt{(n+1)\cdots(n+4)}|n+4\rangle \\
& -\frac{1}{2}(56n^3-84n^2-196n-84) \\
& \times \sqrt{(n+1)(n+2)}|n+2\rangle + \frac{1}{8}\sqrt{n(n-1)\cdots(n-7)}|n-8\rangle \\
& + \frac{1}{6}(8n-20)\sqrt{n(n-1)\cdots(n-5)}|n-6\rangle \\
& + \frac{1}{4}(28n^2-84n+98)\sqrt{n(n-1)\cdots(n-3)}|n-4\rangle \\
& + \frac{1}{2}(56n^3-84n^2-196n-84) \\
& \times \sqrt{n(n-1)}|n-2\rangle + O\lambda^2\Bigg].
\end{aligned}
\tag{26}
$$

We also use the following energy expression from the Rayleigh-Schrödinger perturbation theory

$$
E_n = E_n^0 + \lambda\langle n|\lambda x^8|n\rangle + \lambda^2\sum_{k\neq n}\frac{|\langle k|\lambda x^8|n\rangle|^2}{E_n^0 - E_k^0} + \cdots, \tag{27}
$$

to find

$$
E_n^0 = \hbar\omega_0\left(n+\frac{1}{2}\right), \tag{28}
$$

$$
E_n^1 = (\hbar^4/16m^4\omega_0^4)[70n^4+140n^3+350n^2+280n+105], \tag{29}
$$

$$
\begin{aligned}
E_n^2 = \hbar^7/(256m^8\omega_0^9)[& -426n^7 - 2426n^6 - 47909.5n^5 + 11965.5n^4 - 604092.5n^3 \\
& + 121838.5n^2 - 1282570n - 162960].
\end{aligned}
\tag{30}
$$

4 Classical Limit

It is well known that coherent states are the most classical ones; meaning for example, that the expectation value of the $\hat{x}$-operator in these states, along with application of an appropriate limiting process, does lead to the classical solution [41, 42]. Then, one may expect also that perturbed coherent states defined in an appropriate way, should yield the perturbed classical solution in a similar manner; this is the line of thought that we want to follow for the octic anharmonic oscillator in this section. Thus, we define our initial perturbed coherent states similar to the normal ones, in terms of the perturbed number states as follows

$$|\alpha\rangle_p = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\psi_n\rangle. \tag{31}$$

At time t we write

$$|\alpha, t\rangle_p = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{iE_n t}{\hbar}} |\psi_n\rangle. \tag{32}$$

The expectation value of the $\hat{x}$-operator at time t is given by

$${}_p\langle\alpha, t|x|\alpha, t\rangle_p = \sqrt{\frac{\hbar}{2m\omega_0}} \left\{ \sum_{n=0} \sum_{m=0} e^{-|\alpha|^2} \frac{|\alpha|^n |\alpha|^m}{\sqrt{n!}\sqrt{m!}} e^{-\frac{\text{it}}{\hbar}(E_n - E_m)} \langle\psi_m|a|\psi_n\rangle + c.c \right\}, \tag{33}$$

where "$c.c.$" stands for the complex conjugate of the other terms in the curly bracket. We may assume $\alpha = |\alpha|e^{i\phi}$ and $\phi = 0$ for simplicity, without loss of any generality; in fact our final conclusions do not depend on this specific choice of α. Defining

$$T_{j,k} = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \exp\left[\frac{it}{\hbar}(E_{n+j} - E_{n+k}) \right], \tag{34}$$

and substituting (26) and (34) in (33) we obtain

$$\begin{aligned}
{}_p\langle\alpha, t|x|\alpha, t\rangle_p = {} & \sqrt{\frac{\hbar}{2m\omega_0}} e^{-|\alpha|^2} |\alpha| \big(T_{0,1} + T^*_{0,1}\big) \\
& + \frac{\lambda}{\hbar\omega_0} \left(\frac{\hbar}{2m\omega_0}\right)^4 \sqrt{\frac{\hbar}{2m\omega_0}} e^{-|\alpha|^2} \bigg\{ -\frac{56}{6} |\alpha|^7 \big(T_{6,1} + T^*_{6,1}\big) \\
& - 28|\alpha|^5 \big(T_{5,0} + T^*_{5,0}\big) \\
& - 42|\alpha|^7 \big(T_{5,2} + T^*_{5,2}\big) - 210|\alpha|^5 \big(T_{4,1} + T^*_{4,1}\big) - 210|\alpha|^3 \big(T_{3,0} + T^*_{3,0}\big) \\
& - 140|\alpha|^7 \big(T_{4,3} + T^*_{4,3}\big) \\
& -822|\alpha|^5 \big(T_{3,2} + T^*_{3,2}\big) - 1242|\alpha|^3 \big(T_{2,1} + T^*_{2,1}\big) - 420|\alpha| \big(T_{1,0} + T^*_{1,0}\big) \\
& + 84|\alpha|^7 \big(T_{2,5} + T^*_{2,5}\big) \\
& + 420|\alpha|^5 \big(T_{1,4} + T^*_{1,4}\big) + 420|\alpha|^3 \big(T_{0,3} + T^*_{0,3}\big) + 14|\alpha|^7 \big(T_{1,6} + T^*_{1,6}\big) \\
& + 67|\alpha|^5 \big(T_{0,5} + T^*_{0,5}\big) + \frac{8}{6} |\alpha|^7 \big(T_{0,7} + T^*_{0,7}\big) - |\alpha|^7 \big(T_{7,0} + T^*_{7,0}\big) \bigg\},
\end{aligned} \tag{35}$$

where $T^*_{j,k}$ is the complex conjugate of $T_{j,k}$. Using (28, 29) we may write

$$\begin{aligned}
E_{n+j} - E_{n+k} = {} & \hbar\omega_0(j-k) + \lambda\hbar^4 \big\{ \alpha\big\lfloor (n+j)^4 - (n+k)^4 \big\rfloor + \beta\big\lfloor (n+j)^3 - (n+k)^3 \big\rfloor \\
& + \gamma\big\lfloor (n+j)^2 - (n+k)^2 \big\rfloor + \eta\big[(n+j) - (n+k)\big] \big\},
\end{aligned} \tag{36}$$

where

$$\alpha = \frac{70}{(2m\omega_0)^4}, \qquad \beta = \frac{140}{(2m\omega_0)^4}, \qquad \gamma = \frac{350}{(2m\omega_0)^4}, \qquad \eta = \frac{280}{(2m\omega_0)^4}. \tag{37}$$

Also defining

$$\begin{aligned} a &= 4\alpha, \\ b &= 4(j+k)\alpha + 3\beta, \\ c &= \left\{4\alpha(j^2+k^2+jk) + 3\beta(j+k) + 2\gamma\right\}, \\ d &= \left\{\alpha(j+k)(j^2+k^2) + \beta(j^2+k^2+jk) + \gamma(j+k) + \eta\right\}, \end{aligned} \tag{38}$$

we finally write

$$E_{n+j} - E_{n+k} = \hbar\omega_0(j-k)\left\{1 + \lambda\hbar^3(an^3 + bn^2 + cn + d)\right\}, \tag{39}$$

up to the first order in the parameter λ.

Now, we are ready to find the classical limit of (35). Of course $\hbar \to 0$ limit alone, will not serve our purpose; because everything vanishes as a result. Thus, we also require $\alpha \to \infty$ simultaneously. In fact we keep $\sqrt{\hbar}\alpha$ constant such that

$$\lim_{\substack{\hbar\to 0 \\ |\alpha|\to\infty}} \hbar^{\mathrm{j}} |\alpha|^{2j} = \left(\frac{1}{2} m\omega_0 A'^2\right)^j. \tag{40}$$

All the terms are not as simple as (40), but we have also to deal with the terms like

$$e^{-|\alpha|^2} T_{j,k} = e^{-|\alpha|^2} \sum_{n=0} \frac{|\alpha|^{2n}}{n!} \exp\left(\frac{it}{\hbar}(E_{n+j} - E_{n+k})\right), \tag{41}$$

as well. Assuming small t, and writing the energy terms up to the first order in the parameterλ, we may write

$$e^{-|\alpha|^2} T_{j,k} = e^{-|\alpha|^2} \sum_{n=0} \frac{|\alpha|^{2n}}{n!} \left[\exp(it\omega_0(j-k))\left\{1 + \lambda\hbar^3(an^3 + bn^2 + cn + d)\right\}\right]. \tag{42}$$

Now, using the relations

$$\sum_{n=1}^{\infty} n \frac{|\alpha|^{2n}}{n!} = |\alpha|^2 e^{|\alpha|^2}, \tag{43}$$

$$\sum_{n=1}^{\infty} n^2 \frac{|\alpha|^{2n}}{n!} = \left(|\alpha|^2 + |\alpha|^4\right) e^{|\alpha|^2}, \tag{44}$$

$$\sum_{n=1}^{\infty} n^3 \frac{|\alpha|^{3n}}{n!} = \left(|\alpha|^2 + 3|\alpha|^4 + |\alpha|^6\right) e^{|\alpha|^2}, \tag{45}$$

$$\sum_{n=1}^{\infty} n^4 \frac{|\alpha|^{4n}}{n!} = \left(|\alpha|^2 + 7|\alpha|^4 + 6|\alpha|^6 + |\alpha|^8\right) e^{|\alpha|^2}, \tag{46}$$

and also the limiting criterion (41) we write

$$\lim_{\substack{\hbar\to 0\\ |\alpha|\to\infty}} e^{-|\alpha|^2} T_{j,k} = \exp\big[it\omega_0(j-k)\{1+a\lambda\hbar^3 n^3\}\big] = \exp(i\omega'(j-k)t), \tag{47}$$

where

$$\omega' = \omega_0\left(1+\lambda a\left(\frac{1}{8}m^3\omega_0^3 A'^6\right)+\cdots\right). \tag{48}$$

Using (37, 38) we may also write

$$\omega' = \omega_0\left(1+\lambda\left(\frac{280\mathrm{m}^3\omega_0^3 A'^6}{8\omega_0(2m\omega_0)^4}\right)+\cdots\right) = \omega_0\left(1+\frac{35\lambda \mathrm{A}'^6}{16\mathrm{m}\omega_0^2}+\cdots\right). \tag{49}$$

Now, application of (39), (40), 47) and (49) in (37) leads to the result

$$\lim_{\substack{\hbar\to 0\\ |\alpha|\to\infty}} {}_p\langle\alpha,t|x|\alpha,t\rangle_p = A'\cos\omega' t + \frac{\lambda A'^7}{128m\omega_0^2}\left[\frac{28}{6}\cos 5\omega' t + 42\cos 3\omega' t \right.$$
$$\left. -140\cos\omega' t + \frac{1}{3}\cos 7\omega' t\right]. \tag{50}$$

Finally, considering the following quantum initial conditions, corresponding to the classical initial conditions (14)

$${}_p\langle\alpha,t|x|\alpha,t\rangle_{p,t=0} = A, \qquad {}_p\langle\alpha,t|p|\alpha,t\rangle_{p,t=0} = 0, \tag{51}$$

we find

$$A' = A + \frac{279\lambda}{384m\omega_0^2}A^7, \tag{52}$$

$$\omega' = \omega_0\left(1+\frac{35\lambda}{16m\omega_0^2}A^6+\cdots\right) = \omega, \tag{53}$$

and

$$\lim_{\substack{\hbar\to 0\\ |\alpha|\to\infty}} {}_p\langle\alpha,t|x|\alpha,t\rangle_p = A\cos\omega' t + \frac{\lambda A^7}{128m\omega_0^2}\left[-47\cos\omega' t + 42\cos 3\omega' t \right.$$
$$\left. + \frac{14}{3}\cos 5\omega' t + \frac{1}{3}\cos 7\omega' t\right] + o\lambda^2. \tag{54}$$

The last two results are exactly similar to the classical results (15) and (16).

5 Discussion and Conclusions

We derived the first order classical solution for the octic anharmonic oscillator by means of the Poincaré-Linstedt method. We observed a frequency shift proportional to the sixth power of the amplitude for this system and cosine terms with coefficients proportional to the seventh power of the amplitude with frequencies equal to ω_0, $3\omega_0$, $5\omega_0$ and $7\omega_0$ in the first order solution. Both results are in agreement with those obtained from Taylor series method [33]. We also used the Rayleigh-Schrödinger perturbation theory to find the eigenenergies, eigenstates and the corresponding coherent states for this system. Then, we worked out the expectation value of the position operator and enforced a classical limiting process, along with an appropriate initial condition to retrieve the classical solution. As a result, we have established that the sequence of the Rayleigh-Schrödinger perturbation theory, perturbed coherent states and our limiting process, is consistent with the classical result obtained via the Hamilton's equations of motion and the Poincaré-Linstedt method, for the octic anharmonicity.

A note is in order now. The higher order calculations are tedious and lengthy; that is why we have only worked out the first order calculations in this work. However, the method has been previously applied up to the second order, to the case of other systems with success [34–36]. Moreover, Banerjee and Bhattacharjee [43], also applying multiple-scale perturbation techniques to study the quartic anharmonic oscillators, have observed a smooth cross over to the classical result as $\hbar \to 0$. Encouraged by these observations, we may expect the quantum-classical correspondence to uphold up to higher orders too.

We also note that applying our limiting process we retrieved the classical result from the quantum one, while in reference [37] the reverse pass is taken; the classical solution is the starting point from which the quantum solution is derived. Based on this reciprocal consistency, we may conclude that the question of convergence stands on the same footing in the classical and the quantum domains in this problem; implying that substantiation of the convergence in one of these domains, possibly using Borel summability or Padé approximation methods [44–46], will uphold in both. However, this is an open question that must be tackled with in the future.

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